Expressions for the perfect matching numbers of cubic $l \times m \times n$ lattices and their asymptotic values

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Received 8 January 1996; revised 21 May 1996

A general expression of the perfect matching number for the $l \times m \times n$ cubic lattice was conjectured and examined for infinitely large systems. The asymptotic value of the square of the perfect matching number was calculated by numerical integration. The present treatment will give a key to obtain the true analytic solution of the perfect matching numbers for the 3-dimensional lattices.

1. Introduction

The number of perfect matching for certain kinds of lattices is a key quantity in the theoretical treatments of adsorption of diatomic molecules on metallic surfaces (dimer statistics), nearest-neighbor interaction on the lattice points in anti-ferromagnetic metals (Ising model), and stability of aromatic hydrocarbon molecules (Kekulé structures). Thus the analytical expression for the perfect matching number, K, has been one of the continuous targets in these fields, rather than in the graph theory [1–3].

Especially for the 2-dimensional planar lattice, Kasteleyn [4] and Temperley and Fisher [5,6] independently derived the following beautiful expression:

$$K(2m \times 2n) = 2^{2mn} \prod_{k=1}^{m} \prod_{l=1}^{n} \left[\cos^2\left(\frac{k\pi}{2m+1}\right) + \cos^2\left(\frac{l\pi}{2n+1}\right) \right]. \tag{1}$$

It was shown that for the $m \times n$ torus lattice the K number is given as a linear combination of pfaffians [4].

On the other hand, Hosoya and Ohkami devised an operator technique [7,8] which enables systematic derivation of the recursion formula of the perfect matching numbers for periodic lattices. Hock and McQuistan obtained the recursion formula of various series of graphs [9,10].

By the use of the results obtained by the operator technique and the method of determinant given by Kasteleyn the present authors derived the expressions for the perfect matching numbers of $2 \times 2 \times n$ [11] and $2 \times 3 \times n$ cubic lattices [12]:

$$K(2 \times 2 \times n) = \prod_{j=1}^{n} 2^2 \left[\cos^2\left(\frac{\pi}{3}\right) + \cos^2\left(\frac{\pi}{3}\right) + \cos^2\left(\frac{j\pi}{n+1}\right)\right], \tag{2}$$

$$K(2 \times 3 \times n) = \sum_{j=1}^{5} C_j (\det \tilde{D}_{n,j})^{1/4} \Big/ \prod_{j=1}^{n} \left\{1 + 4\cos^2\left(\frac{k\pi}{n+1}\right)\right\}^{1/4}, \tag{3}$$

where C_j's are constants, and the matrix $\tilde{D}_{n,j}$ is obtained by diagonalizing the matrix $D_{n,j}$, which shows the adjacency relation among the lattice points.

By the same method they predicted the general expression and proved some of the expressions for the $\overline{m \times n}$ cylindrical lattices [13]:

$$K(\overline{2m \times n}) = \prod_{j=1}^{[n/2]} \prod_{k=1}^{m} \left[4\left(\cos^2 \frac{k\pi}{2m+1} + \sin^2 \frac{(2j-1)\pi}{n}\right)\right] \tag{4}$$

(see ref. [13] for the case of $K((\overline{2m-1) \times n}))$).

After generalizing the expression for the perfect matching number of $2 \times 3 \times n$ lattices the expression for that of $2 \times m \times n$ lattices was conjectured as follows (see ref. [14]):

$$K(2 \times m \times n) = \sum_{b=1}^{b'} \sum_{c=1}^{c'} k_{bc} (\det \tilde{D}_{2,m,n,b,c})^{\epsilon} / \mathrm{g}(m)\mathrm{h}(n)\,, \tag{5}$$

where k_{bc} and ϵ are constants to be determined by the prescribed method. The quantities $\mathrm{g}(m)$ and $\mathrm{h}(n)$ are expressed as

$$\mathrm{g}(m) \equiv \prod_{g=1}^{m} (u_1^2 x^2 + v_1^2 y^2 + w_1^2 z^2 \cos^2[g\pi/(m+1)])^{\beta}\,,$$

$$\mathrm{h}(n) \equiv \prod_{h=1}^{n} (u_2^2 x^2 + v_2^2 y^2 + w_2^2 z^2 \cos^2[h\pi/(n+1)])^{\gamma}\,,$$

where β, γ, u_i, v_i and w_i $(i = 1, 2)$ are constants.

In the limit when m and n respectively approach to infinity the asymptotic form of eq. (5) is obtained as

$$\ln K = \frac{1}{\pi} \int_0^{\pi/2} \ln\left(\left[\frac{x^2}{4} + z_C^2 \cos^2\phi\right]^{1/2} + \left[y_B^2 + \frac{x^2}{4} + z_C^2 \cos^2\phi\right]^{1/2}\right) d\phi\,, \quad \epsilon = 1/4\,, \tag{6}$$

where x, y_B and z_C are quantities which show the adjacency relation between the lattice points. It was shown in ref. [14] that one of the special expressions of eq. (6) becomes the same equation as the one obtained by Kasteleyn [4].

The agreement of one of the special solutions of the expression in eq. (6) with that obtained by Kasteleyn suggests the correctness of the present calculation for the $l \times m \times n$ lattice, because the calculation for the $l \times m \times n$ lattice is a similar extention of the case for the $2 \times m \times n$ lattice.

The aim of the present paper is to search a key for obtaining the true expressions for the perfect matching numbers of cubic $l \times m \times n$ lattices by expanding the assumption such as eq. (5) to the one suitable to the $l \times m \times n$ lattice and to obtain an asymptotic form such as eq. (6), because the analytical treatment of dimer model on cubic lattices has not yet been solved successfully.

2. Perfect matching numbers

Each lattice point p on the $l \times m \times n$ cubic lattice (Fig. 1) is expressed by the coordinates (i,j,k) as follows:

$$(i,j,k) \leftrightarrow p = i + (j-1)l + (k-1)lm\,. \tag{7}$$

Examples of numbering of $2 \times 2 \times n$ lattices and $2 \times 3 \times n$ lattices are shown in refs. [11] and [12], respectively.

The lattice can be covered by $lmn/2$ dimers in the canonical order:

$$C = |p_1;p_2|\,|p_3;p_4|\,|p_5;p_6|\cdots, \tag{8}$$

where $l =$ even, and

$$p_1 < p_2 < p_3 < p_4;\ldots;p_{lmn-1} < p_{lmn}\,. \tag{9}$$

The adjacency relation among these lattice points are expressed as follows:

$$D_{l,m,n,a,b,c} = x_a Q_l \otimes E_m \otimes E_n + y_b F_l \otimes Q_m \otimes E_n + z'_c E_l \otimes E_m \otimes Q_n\,, \tag{10}$$

where $\otimes$ means a direct product of matrices, and the quantities x_a, y_b and z'_c signify the bonding between two lattice points.

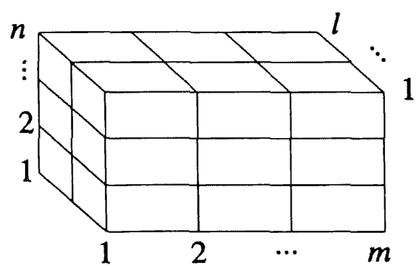

Fig. 1. $l \times m \times n$ cubic lattice.

Matrices Q_n and F_n were used by Kasteleyn et al. and have the expressions as shown below; E_n is a unit matrix of order n [4,11–13].

$$F_n \equiv \begin{pmatrix} -1 & 0 & 0 & 0 & & \\ 0 & 1 & 0 & 0 & & \\ 0 & 0 & -1 & 0 & & \\ 0 & 0 & 0 & 1 & & \\ & & & & \ddots & \\ & & & & & (-1)^n \end{pmatrix},$$

$$Q_n \equiv \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ -1 & 0 & 1 & 0 & \\ 0 & -1 & 0 & 1 & \\ \vdots & & & \ddots & \\ 0 & & & & 0 \end{pmatrix}.$$

The matrix $D_{l,m,n,a,b,c}$ is diagonalized by U_n^{-1} and U_n, where

$$U_n(j,k) = \sqrt{\frac{2}{n+1}} i^j \sin\frac{jk\pi}{n+1}, \tag{11}$$

$$U_n^{-1}(j,k) = \sqrt{\frac{2}{n+1}} (-i)^k \sin\frac{jk\pi}{n+1}, \tag{12}$$

$$\begin{aligned} \tilde{D}_{l,m,n,a,b,c} &\equiv U_l^{-1} \otimes U_m^{-1} \otimes U_n^{-1} D_{l,m,n,a,b,c} U_l \otimes U_m \otimes U_n \\ &= x_a U_l^{-1} Q_l U_l \otimes U_m^{-1} E_m U_m \otimes U_n^{-1} E_n U_n \\ &\quad + y_b U_l^{-1} F_l U_l \otimes U_m^{-1} Q_m U_m \otimes U_n^{-1} E_n U_n \\ &\quad + z'_c U_l^{-1} E_l U_l \otimes U_m^{-1} E_m U_m \otimes U_n^{-1} Q_n U_n . \end{aligned} \tag{13}$$

The eigenvalues of matrices Q_l, Q_m and Q_n are denoted as λ_f, μ_g and ν_h, respectively:

$$\lambda_f = 2i\cos[f\pi/(l+1)], \tag{14}$$

$$\mu_g = 2i\cos[g\pi/(m+1)], \tag{15}$$

$$\nu_h = 2i\cos[h\pi/(n+1)]. \tag{16}$$

The diagonalized matrix $\tilde{D}_{l,m,n,a,b,c}$ is expressed by the use of the above eigenvalues as

$$\tilde{D}_{l,m,n,a,b,c} = \prod_{g=1}^{m} \begin{vmatrix} O_{1,1} & & & -M_g \\ & O_{2,1} & -M_g & \\ & & \ddots & \\ -M_g & & & O_{l,1} \end{vmatrix} \cdot \prod_{g=1}^{m} \begin{vmatrix} O_{1,2} & & & -M_g \\ & O_{2,2} & -M_g & \\ & & \ddots & \\ -M_g & & & O_{l,2} \end{vmatrix} \cdots$$

$$\cdots \prod_{g=1}^{m} \begin{vmatrix} O_{1,n} & & & -M_g \\ & O_{2,n} & -M_g & \\ & & \ddots & \\ -M_g & & & O_{l,n} \end{vmatrix} ,$$

where

$$\Lambda_f \equiv x_a \lambda_f \qquad (f = 1, 2, \ldots, l) ,$$

$$M_g \equiv y_b \mu_g \qquad (g = 1, 2, \ldots, m) ,$$

$$N_h \equiv z'_c \nu_h \qquad (h = 1, 2, \ldots, n) ,$$

$$O_{i,j} \equiv \Lambda_i + N_j .$$

The above matrix $\tilde{D}_{l,m,n,a,b,c}$ is reduced to

$$\tilde{D}_{l,m,n,a,b,c} = \prod_{g=1}^{m} \prod_{h=1}^{n} \begin{vmatrix} O_{1,h} & & & -M_g \\ & O_{2,h} & -M_g & \\ & & \ddots & \\ -M_g & & & O_{l,h} \end{vmatrix} . \tag{17}$$

The determinant following the two product signs is decomposed as follows (see eq. (23) in ref. [13]):

$$(O_{1,h} O_{2L,h} - M_g^2)(O_{2,h} O_{2L-1,h} - M_g^2) \cdots (O_{L,h} O_{L+1,h} - M_g^2) ,$$

where $l \equiv 2L$ (L is a positive integer). From this the following expression is obtained:

$$\det \tilde{D}_{l,m,n,a,b,c} = \prod_{f=1}^{l/2} \prod_{g=1}^{m} \prod_{h=1}^{n} (O_{f,h} O_{f',h} - M_g^2) , \tag{18}$$

where $f' \equiv 2L - f + 1 = l - f + 1$.

By the use of the definition

$$O_{f,h} = \Lambda_f + N_h = x_a \lambda_f + z'_c \nu_h ,$$

$O_{f,h}O_{f',h}$ is calculated as

$$O_{f,h}O_{f',h} = x_a^2\lambda_f\lambda_{f'} + z_c'^2\nu_h^2 + x_a z_c'\nu_h(\lambda_f + \lambda_{f'}) \,. \tag{19}$$

From the equations

$$\lambda_f + \lambda_{f'} = 0 \tag{20}$$

and

$$\lambda_f\lambda_{f'} = 4\cos^2[f\pi/(l+1)] \,, \tag{21}$$

together with eq. (16), one can obtain

$$O_{f,h}O_{f',h} = 4x_a^2\cos^2[f\pi/(l+1)] - 4z_c'^2\cos^2[h\pi/(n+1)] \,.$$

Then the determinant can be expressed as

$$\det \tilde{D}_{l,m,n,a,b,c} = \prod_{f=1}^{l/2}\prod_{g=1}^{m}\prod_{h=1}^{n}(4x_a^2\cos^2[f\pi/(l+1)] + 4y_b^2\cos^2[g\pi/(m+1)] + 4z_c^2\cos^2[h\pi/(n+1)]) \,, \tag{22}$$

where $z_c^2 \equiv -z_c'^2 > 0$.

According to the case of $2 \times 3 \times n$ lattices, where the rigorous solution was given, the expression of the perfect matching number for $l \times m \times n$ lattices is conjectured to be [14]

$$K(l \times m \times n) = \sum_{a=1}^{a'}\sum_{b=1}^{b'}\sum_{c=1}^{c'} k_{abc} \prod_{f=1}^{l'}\prod_{g=1}^{m'}\prod_{h=1}^{n'}\{4x_a^2\cos^2[f\pi/(l'+1)] + 4y_b^2\cos^2[g\pi/(m'+1)] + 4z_c^2\cos^2[h\pi/(n'+1)]\}^{\epsilon} \,, \tag{23}$$

where $l' = k_1 l$, $m' = k_2 m$ and $n' = k_3 n$, and ϵ, k_1, k_2 and k_3 are constants to be determined.

The assumption of eq. (23) is slightly different from the case of the $(2 \times m \times n)$ lattice. That is, the denominator was deleted and the indices l, m and n were scaled to l', m' and n'.

An example is shown as follows:

$$K(2 \times 3 \times 4) = \sum_{a=1}^{a'=1}\sum_{b=1}^{b'=5}\sum_{c=1}^{c'=1} k_{abc} \prod_{f=1}^{l'=2}\prod_{g=1}^{m'=2}\prod_{h=1}^{n'=4}\{\cdots\}^{\epsilon}$$

$$= \sum_{b=1}^{5} k_b \left\{ \prod_{h=1}^{4} (4x_a^2 \cos^2[\pi/3] + 4y_b^2 \cos^2[\pi/3] + 4z_c^2 \cos^2[h\pi/5])^{1/8} \right.$$

$$\times \prod_{h=1}^{4} (4x_a^2 \cos^2[2\pi/3] + 4y_b^2 \cos^2[\pi/3] + 4z_c^2 \cos^2[h\pi/5])^{1/8}$$

$$\times \prod_{h=1}^{4} (4x_a^2 \cos^2[\pi/3] + 4y_b^2 \cos^2[2\pi/3] + 4z_c^2 \cos^2[h\pi/5])^{1/8}$$

$$\left. \times \prod_{h=1}^{4} (4x_a^2 \cos^2[2\pi/3] + 4y_b^2 \cos^2[2\pi/3] + 4z_c^2 \cos^2[h\pi/5])^{1/8} \right\} .$$

If $x_a^2 = x^2, y_b^2 = 2y_j^2, z_c^2 = -z^2$, and

$$k_b = k_j = \sqrt{-A_j\overline{A_j}}(\overline{Q_j} - Q_j) ,$$

then eq. (28) for $n = 4$ in ref. [12] is obtained.

Further study of the expression for perfect matching numbers for the $2 \times 4 \times n$, $2 \times 5 \times n$, and other lattices might decrease the number of summations $\sum$ in eq. (23). However, the conclusion of the present treatment is unchanged.

Calculating $K_2 \equiv \lim_{m,n\to\infty} K_{2,m,n}^{1/mn}$, the following expression for the asymptotic value is obtained:

$$\ln K_2 = 2\epsilon \ln y_B + \frac{4\epsilon}{\pi} \int_0^{\pi/2} \ln([\overline{\xi^2} + \eta^2 \cos^2\phi]^{1/2} + [1 + \overline{\xi^2} + \eta^2 \cos^2\phi]^{1/2}) d\phi , \qquad (24)$$

where $\overline{\xi^2} \equiv x^2/4y_B^2$ and $\eta^2 \equiv z_C^2/y_B^2$.

Substituting $\xi^2 = 0$ and $\epsilon = 1/4$ into eq. (24) the same expression was obtained as by Kasteleyn [4]. (See the equation just below eq. (17) in ref. [4].)

3. Asymptotic value of the perfect matching number K

Let us put the maximum values of the coefficients x_a^2, y_b^2 and z_c^2 in eq. (23) as x_A^2, y_B^2 and z_C^2, respectively.

The expression $K_{l,m,n}$ in eq. (23) can be changed by the use of the quantities x_A^2, y_B^2 and z_C^2 to

$$K_{l,m,n} = \left[\prod_{f=1}^{l'/2} \prod_{g=1}^{m'/2} \prod_{h=1}^{n'} \sigma_{\max} \left\{ k_{ABC} + \sum_{a\neq A} \sum_{b\neq B} \sum_{c\neq C} k_{abc} \prod_{f=1}^{l'/2} \prod_{g=1}^{m'/2} \prod_{h=1}^{n'} \frac{\sigma}{\sigma_{\max}} \right\} \right]^{2\epsilon} , \qquad (25)$$

where

$$\sigma \equiv 4x_a^2 \cos^2[f\pi/(l'+1)] + 4y_b^2 \cos^2[g\pi/(m'+1)] + 4z_c^2 \cos^2[h\pi/(n'+1)] , \tag{26}$$

and

$$\sigma_{\max} \equiv 4x_A^2 \cos^2[f\pi/(l'+1)] + 4y_B^2 \cos^2[g\pi/(m'+1)] + 4z_C^2 \cos^2[h\pi/(n'+1)] . \tag{27}$$

Since the following expression has been established:

$$\lim_{l,m,n\to\infty} \prod_f \prod_g \prod_h \sigma/\sigma_{\max} = 0 ,$$

the parentheses in eq. (23) become

$$\lim_{l,m,n\to\infty} \{ \quad \}^{2\epsilon/lmn} = 1 . \tag{28}$$

Therefore, the perfect matching number K per lattice point can be shown to be

$$\begin{aligned} K &\equiv \lim_{l,m,n\to\infty} K^{1/lmn} \\ &= \lim_{l,m,n\to\infty} \left(\prod_{f=1}^{1/2} \prod_{g=1}^{m/2} \prod_{h=1}^{n} \sigma_{\max} \right)^{2\epsilon/lmn} \\ &\equiv \lim_{l,m,n\to\infty} \overline{K}_{l,m,n}^{1/lmn} \end{aligned} \tag{29}$$

The expression for $\overline{K}_{l,m,n}$ is given by

$$\begin{aligned} \overline{K}_{l,m,n} = \Bigg\{ y_B^{lmn/2} \prod_{f=1}^{l/2} \prod_{g=1}^{m/2} \prod_{h=1}^{n} & \times 4(\xi^2 \cos^2[f\pi/(l+1)] + \cos^2[g\pi/(m+1)] \\ & + \eta^2 \cos^2[h\pi/(n+1)]) \Bigg\}^{2\epsilon} , \end{aligned} \tag{30}$$

where

$$\xi^2 \equiv x_A^2/y_B^2 \quad \text{and} \quad \eta^2 \equiv z_C^2/y_B^2 .$$

Applying the following identity to eq. (30) (see eq. (14) in ref. [4]):

$$\prod_{g=1}^{m/2} 4\left[u^2 + \cos^2 \frac{g\pi}{m+1}\right] = \frac{[u + (1+u^2)^{1/2}]^{m+1} - [u - (1+u^2)^{1/2}]^{m+1}}{2(1+u^2)^{1/2}} , \tag{31}$$

the expression for $\overline{K}_{l,m,n}$ becomes

$$\overline{K}_{l,m,n} = \prod_{f=1}^{l/2} \prod_{h=1}^{n} \frac{[u + (1+u^2)^{1/2}]^{m+1} - [u - (1+u^2)^{1/2}]^{m+1}}{2(1+u^2)^{1/2}}, \tag{32}$$

where

$$u^2 \equiv \xi^2 \cos^2[f\pi/(l+1)] + \eta^2 \cos^2[h\pi/(n+1)].$$

By the use of eq. (32), the asymptotic value of $\overline{K}_{l,m,n}$ in the limit of $m \to \infty$ becomes

$$\begin{aligned} K_{ln} &\equiv \lim_{m\to\infty} \overline{K}_{lmn}^{1/m} \\ &= \left[y_B^{ln/2} \prod_{f=1}^{l/2} \prod_{h=1}^{n} \{u + (1+u^2)^{1/2} \right]^{2\epsilon}. \end{aligned} \tag{33}$$

From eq. (33) the logarithm of K_{ln} is given by

$$\ln K_{ln} = nl\epsilon \ln y_B + \ln \left[\prod_{f=1}^{l} \prod_{h=1}^{n} \{u + (1+u^2)^{1/2}\}^{1/2} \right]^{2\epsilon};$$

namely, we have

$$\ln K_{ln}^{1/l} = n\epsilon \ln y_B + \frac{\epsilon}{l} \sum_{f=1}^{l} \ln \prod_{h=1}^{n} \{u + (1+u^2)^{1/2}\}^{1/2}. \tag{34}$$

The asymptotic value of $K_{ln}^{1/l}$ in the limit of $l \to \infty$ can be expressed as an integral as follows:

$$\begin{aligned} \ln K_{ln} &\equiv \lim_{n\to\infty} \ln K_{ln}^{1/l} \\ &= n\epsilon \ln y_B + \frac{\epsilon}{\pi} \int_0^{\pi} \sum_{h=1}^{n} \ln[\{\xi^2 \cos^2\phi + \eta^2 \cos^2[h\pi/(n+1)]\}^{1/2} \\ &\quad + \{1 + \xi^2 \cos^2\phi + \eta^2 \cos^2[h\pi/(n+1)]\}^{1/2}] d\phi. \end{aligned} \tag{35}$$

Moreover, the asymptotic value of $\ln K_n$ in the limit of $n \to \infty$ is obtained as

$$\begin{aligned} \ln K &\equiv \lim_{n\to\infty} \ln K_{ln}^{1/n} \\ &= \epsilon \ln y_B + \frac{4\epsilon}{\pi^2} \int_0^{\pi/2} \int_0^{\pi/2} \ln[\{\xi^2 \cos^2\phi + \eta^2 \cos^2\theta\}^{1/2} \\ &\quad + \{1 + \xi^2 \cos^2\phi + \eta^2 \cos^2\theta\}^{1/2}] d\theta d\phi. \end{aligned} \tag{36}$$

Eq. (36) can also be shown as

$$\ln K = \frac{4\epsilon}{\pi^2} \int_0^{\pi/2} \int_0^{\pi/2} \ln[\{x_A^2 \cos^2 \phi + Z_C^2 \cos^2 \theta\}^{1/2} + \{y_B^2 + x_A^2 \cos^2 \phi + z_C^2 \cos^2 \theta\}^{1/2}] d\theta d\phi\,. \tag{37}$$

Eqs. (36) or (37) is the limit of the configurational partition function for one lattice point of the $l \times m \times n$ lattice.

Substituting $x_A = y_B = z_C = 1$ into eq. (36) or (37), the limit of the number of dimer arrangements $g(lmn/2)$ can be calculated (see ref. [4]).

The "molecular freedom" φ_2 which is defined as the number of arrangements per dimer is expressed as

$$\varphi_2 = \{g(lmn/2)\}^{2/lmn} = \{K_{lmn}(x_A = y_B = z_C = 1)\}^{2/lmn}\,. \tag{38}$$

In the lattice of the limit of l, m and n ($l, m, n \to \infty$), the asymptotic value of φ_2, i.e., $\varphi_2^{(\infty)}$ is

$$\varphi_2^{(\infty)} = \{K(x_A = y_B = z_C = 1)\}^2 = 1.519448336\ldots\,. \tag{39}$$

The K number is 9, for example, for the case of the smallest cubic lattice $2 \times 2 \times 2$. Therefore,

$$\varphi_2 = (9)^{2/2\times2\times2} = \sqrt{3} = 1.732\cdots$$

in this case (see ref. [3] for the values of K in the case of various polycube lattices).

The above values are somewhat smaller than the same kind of value, 1.791622812..., for the $m \times n$ lattice (see ref. [4]).

The value of the limit $\varphi_2^{(\infty)}$ was obtained by the numerical integral of the following equation:

$$\ln K(x_A = y_B = z_C = 1, \epsilon = 1/4) = \frac{1}{\pi^2} \int_0^{\pi/2} \int_0^{\pi/2} \ln[(\cos^2 \phi + \cos^2 \theta)^{1/2} + (1 + \cos^2 \phi + \cos^2 \theta)^{1/2}] d\theta d\phi\,. \tag{40}$$

The constant value $\epsilon = 1/4$ was obtained according to the case of $2 \times m \times n$ lattice.

The next task is to find other keys for obtaining the true expression for the perfect matching numbers of 3-dimensional lattices and also for solving the 3-dimensional Ising model [15,16] on the basis of the method described above.

Acknowledgement

One of authors (HN) is grateful to Dr. Hiroshi Murakami for valuable discussions and for calculating numerical integrals.

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